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Asymptotics of skew orthogonal polynomials

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Abstract
Exact integral expressions of the skew orthogonal polynomials involved in orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) random matrix ensembles are obtained: the (even rank) skew orthogonal polynomials are average characteristic polynomials of random matrices. From there, asymptotics of the skew orthogonal polynomials are derived.

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1. Introduction

Families of orthogonal (or skew orthogonal) polynomials, have many applications to mathematics and physics [1, 2].

Here, we recall applications to random matrix theory (RMT) [3–7], i.e. disordered solid state physics [4], QCD [7] or statistical physics on a random fluctuating lattice [5, 8] (2D quantum gravity). In all these fields of physics, one is interested in the spectrum of a matrix (Hamiltonian, transmission matrix, S-matrix, Dirac operator and so on), which can be considered as random for various reasons (disorder, random impurities, quantum fluctuations, chaos or non-integrability, etc). It was observed that the spectrum of a large random matrix shows universal properties [9, 10] (two-point correlation function, in the short- or long-range regime; universal conductance fluctuations of mesoscopic conductors). One possible way to understand and prove that universality is through the ‘orthogonal polynomials’ method, which we shall recall below. In order to extract some useful numerical results, it is important to have some asymptotics of the orthogonal polynomials in some special limit.

The type of orthogonal polynomials involved depends on the symmetry of the matrix ensemble [3]. The case of a physical system with broken time-reversibility (for instance a mesoscopic conductor in the presence of a magnetic field), represented by a $U(N)$ invariant matrix ensemble, was extensively studied, because it is the simplest [2].

Here, we shall focus on the $O(N)$ and $Sp(2N)$ invariant matrix ensembles, which appear for physical systems with time-reversibility and/or half-integer spin with broken rotational symmetry. These ensembles involve families of skew orthogonal polynomials.

The aim of this article is to present a remarkable exact expression of the skew orthogonal polynomial as an integral, and deduce the required asymptotics from it.

Section 2 is a brief introduction to the orthogonal polynomial's method in RMT, in section 3 we give and prove the remarkable exact expressions for the skew orthogonal polynomials, and in section 4, we consider their asymptotics.

2. The orthogonal polynomials

Consider the partition function of a random matrix M:

$$Z_N^{(\beta)}[V] = \int_{M\in E_N^{(\beta)}} \mathrm{d}M\ \mathrm{e}^{-N_\beta \operatorname{tr} V(M)} \tag{2.1}$$

where $E_N^{(1)}$ is the set of all $N \times N$ real symmetric matrices, $E_N^{(2)}$ is the set of all $N \times N$ hermitian matrices, $E_N^{(4)}$ is the set of all $2N \times 2N$ self-adjoint real quaternionic matrices[1] (see appendix A1) and $\mathrm{d}M$ is the product of Lebesgue measures of all independent real components of the matrix $M \in E_N^{(\beta)}$. $V(x)$ is a polynomial (the potential), bounded from below, and $N_\beta = N, N, N/2$ respectively for $\beta = 1, 2, 4$.

The angular degrees of freedom of M can be integrated out, and (2.1) can be rewritten as an integral over the N eigenvalues $(\lambda_1, \ldots, \lambda_N)$ of M only [3, 11] :

$$Z_N^{(\beta)}[V] = U_N^{(\beta)} \int |\Delta(\lambda)|^\beta \prod_{i=1}^N \bar{\mathrm{d}}\lambda_i \tag{2.2}$$

where $U_N^{(\beta)}$ is the volume of the group $O(N)$, $U(N)$ or $Sp(2N)$ respectively for $\beta = 1, 2, 4$. $\bar{\mathrm{d}}\lambda = \mathrm{d}\lambda\, \mathrm{e}^{-NV(\lambda)}$ is the measure element, and

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \tag{2.3}$$

is the Vandermonde determinant, which can be rewritten as

$$\Delta(\lambda) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{N-1} \\ \vdots & & & & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \ldots & \lambda_N^{N-1} \end{pmatrix} = \det\left(\lambda_i^j\right) = \det P_j(\lambda_i) \tag{2.4}$$

where $P_j(\lambda) = \lambda^j + \cdots$ is an arbitrary monic polynomial of degree j. The last equality is obtained by linearly mixing columns of the determinant, and the first equality is the well-known Vandermonde determinant, which can be found in any math textbook [3].

The computation of integral (2.2) becomes easier with a special choice of the polynomials $P_j(\lambda)$, chosen orthogonal with respect to an appropriate scalar product [11]:

- In the unitary case $\beta = 2$, the scalar product under consideration is

$$\langle f|g\rangle = \int_{-\infty}^{\infty} \bar{\mathrm{d}}x f(x) g(x) \tag{2.5}$$

[1] $M \in E_N^{(4)}$ can be viewed either as a $2N \times 2N$ matrix with complex number entries or an $N \times N$ block matrix with quaternion entries (which are 2×2 matrices). It has N eigenvalues, each degenerated twice [3].

and the polynomials $P_n(x)$ are chosen orthogonal

$$\langle P_n|P_m\rangle = h_n\delta_{nm}. \tag{2.6}$$

- In the orthogonal case $\beta = 1$, the scalar product under consideration is skew-symmetric

$$\langle f|g\rangle = -\langle g|f\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{\mathrm{d}}x\,\bar{\mathrm{d}}y\ \ f(x)\,\mathrm{sgn}\,(x-y)\ g(y) \tag{2.7}$$

and the polynomials $P_n(x)$ are chosen skew orthogonal

$$\langle P_{2n}|P_{2m}\rangle = \langle P_{2n+1}|P_{2m+1}\rangle = 0 \tag{2.8}$$

$$\langle P_{2n+1}|P_{2m}\rangle = h_n\delta_{nm}. \tag{2.9}$$

- In the symplectic case $\beta = 4$, the scalar product under consideration is skew-symmetric too

$$\langle f|g\rangle = -\langle g|f\rangle = \int_{-\infty}^{\infty} \bar{\mathrm{d}}x\ (f(x)g'(x) - f'(x)g(x)) \tag{2.10}$$

and the polynomials $P_n(x)$ are chosen skew orthogonal

$$\langle P_{2n}|P_{2m}\rangle = \langle P_{2n+1}|P_{2m+1}\rangle = 0 \tag{2.11}$$

$$\langle P_{2n+1}|P_{2m}\rangle = h_n\delta_{nm}. \tag{2.12}$$

In all three cases, the partition function (2.2) reduces mainly[2] to $Z = \prod_{n=1}^{n_{\mathrm{F}}} h_{n-1}$, where n_{F} is called the 'Fermi level' by analogy with a system of fermions

$$n_{\mathrm{F}} = \frac{\beta}{2}N_\beta \qquad \text{for} \quad \beta = 1, 2, 4. \tag{2.13}$$

According to the Christoffel–Darboux theorem and its generalization to skew orthogonal polynomials [17, 18], when N is large, the statistical observables (the correlation functions) are related to properties of the polynomials P_n, with n in the vicinity of the 'Fermi level': $n \to \infty$, $N \to \infty$ and $n - n_{\mathrm{F}} \sim O(1)$.

2.1. Determination of the orthogonal polynomials

For a generic potential $V(x)$, these orthogonal polynomials exist, and can be constructed by recurrence. Indeed, we start from $P_0(x) = 1$, then the coefficients of P_1 are determined by the orthogonality conditions, and by recurrence, we determine P_n and h_n for all n.

Note that for the skew orthogonal polynomials, there is an ambiguity: P_{2n+1} is defined only up to an arbitrary linear combination with P_{2n}. If one wants a unique definition, an extra condition should be added, for instance that the term of degree $2n$ in P_{2n+1} vanishes. Anyway, the values of h_n do not depend on this ambiguity.

The determination of the orthogonal polynomials by recurrence is inefficient if one wants to compute P_n for n large. The aim of this article is to present a closed expression of P_n for any n, and to derive from it some asymptotics in the large n limit, particularly near the Fermi level $n - n_{\mathrm{F}} \sim O(1)$.

[2] The actual result may depend on the parity of N. Details can be found in [3].

3. An exact expression of the skew orthogonal polynomials

- In the unitary case $\beta = 2$, it is known that

$$P_n^{(2)}(x) = \frac{1}{Z_n^{(2)}} \int_{M\in E_n^{(2)}} \mathrm{d}M\, \det(x\mathbf{I} - M)\mathrm{e}^{-N\,\mathrm{tr}\,V(M)}$$
$$= \frac{U_n^{(2)}}{Z_n^{(2)}} \int \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_n \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_i (x - \lambda_i) \qquad (3.14)$$

where $\mathbf{I}$ is the dimension n identity matrix. In other words, the nth orthogonal polynomial is the average of the characteristic polynomial of a $n \times n$ hermitian matrix with respect to the weight $\mathrm{e}^{-N\,\mathrm{tr}\,V(M)}$:

$$P_n^{(2)}(x) = \langle \det(x\mathbf{I} - M)\rangle_{n\times n} \qquad (3.15)$$

This has been known for more than a century [12] (in the context of RMT, see e.g. [1, 2, 13]). We are now going to generalize this expression to $\beta = 1$ and 4.

- Orthogonal case $\beta = 1$. We will prove below that

$$P_{2n}^{(1)}(x) = \frac{1}{Z_{2n}^{(1)}} \int_{M\in E_{2n}^{(1)}} \mathrm{d}M\, \det(x\mathbf{I} - M)\, \mathrm{e}^{-N\,\mathrm{tr}\,V(M)}$$
$$= \frac{U_{2n}^{(1)}}{Z_{2n}^{(1)}} \int \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_{2n} \prod_{i<j} |\lambda_i - \lambda_j| \prod_i (x - \lambda_i)$$
$$= \langle \det(x\mathbf{I} - M)\rangle_{2n\times 2n} \qquad (3.16)$$

and

$$P_{2n+1}^{(1)}(x) = \frac{1}{Z_{2n}^{(1)}} \int_{M\in E_{2n}^{(1)}} \mathrm{d}M\, (x + \mathrm{tr}\, M + c_n) \det(x\mathbf{I} - M)\, \mathrm{e}^{-N\,\mathrm{tr} V(M)}$$
$$= \frac{U_{2n}^{(1)}}{Z_{2n}^{(1)}} \int \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_{2n} \prod_{i<j} |\lambda_i - \lambda_j| \left(x + \sum_i \lambda_i + c_n\right) \prod_i (x - \lambda_i)$$
$$= \langle (x + \mathrm{tr}\, M + c_n) \det(x\mathrm{I} - M)\rangle_{2n\times 2n} \qquad (3.17)$$

the constants c_n can be chosen arbitrarily, the choice $c_n = 0$ being such that the term of degree $2n$ in P_{2n+1} vanishes. $\mathbf{I}$ is the dimension $2n$ identity matrix here.

- Symplectic case $\beta = 4$

$$P_{2n}^{(4)}(x) = \frac{1}{Z_n^{(4)}} \int_{M\in E_n^{(4)}} \mathrm{d}M\, \det(x\mathbf{I} - M)\, \mathrm{e}^{-\frac{N}{2}\,\mathrm{tr} V(M)}$$
$$= \frac{U_n^{(4)}}{Z_n^{(4)}} \int \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_n \prod_{i<j} |\lambda_i - \lambda_j|^4 \prod_i (x - \lambda_i)^2$$
$$= \langle \det(x\mathbf{I} - M)\rangle_{2n\times 2n} \qquad (3.18)$$

and

$$P_{2n+1}^{(4)}(x) = \frac{1}{Z_n^{(4)}} \int_{M\in E_n^{(4)}} \mathrm{d}M\, (x + \mathrm{tr} M + c_n) \det(x\mathbf{I} - M)\, \mathrm{e}^{-\frac{N}{2}\,\mathrm{tr} V(M)}$$
$$= \frac{U_n^{(4)}}{Z_n^{(4)}} \int \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_n \prod_{i<j} |\lambda_i - \lambda_j|^4 \left(x + 2\sum_i \lambda_i + c_n\right) \prod_i (x - \lambda_i)^2$$
$$= \langle (x + \mathrm{tr} M + c_n) \det(x\mathbf{I} - M)\rangle_{2n\times 2n}. \qquad (3.19)$$

$\mathbf{I}$ is the dimension $2n$ identity matrix here (see appendix A1 for a definition of $M \in E_n^{(4)}$).

3.1. Proof of (3.16)

Note that it is sufficient to prove that

$$\left\langle P_{2n} \middle| x^m \right\rangle = 0 \qquad \text{and} \qquad \left\langle P_{2n+1} \middle| x^m \right\rangle = 0 \qquad \text{for all} \quad m \leqslant 2n-1. \tag{3.20}$$

Consider

$$\left\langle P_{2n} \middle| x^m \right\rangle \propto \int \bar{\mathrm{d}}x\ \bar{\mathrm{d}}y\ \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_{2n} \prod_{i<j}(\lambda_i - \lambda_j) \prod_i (x-\lambda_i)$$
$$\times \prod_{i<j} \operatorname{sgn}(\lambda_i - \lambda_j)\ \operatorname{sgn}(x-y)\ \ y^m \tag{3.21}$$

then write $x = \lambda_{2n+1}$:

$$\left\langle P_{2n} \middle| x^m \right\rangle \propto \int \bar{\mathrm{d}}y\ \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_{2n+1}\ \ y^m \prod_{1\leqslant i<j\leqslant 2n+1} (\lambda_i - \lambda_j)$$
$$\times \prod_{1\leqslant i<j\leqslant 2n+1} \operatorname{sgn}(\lambda_i - \lambda_j) \prod_{i=1}^{2n} \operatorname{sgn}(\lambda_i - \lambda_{2n+1})\ \operatorname{sgn}(\lambda_{2n+1} - y) \tag{3.22}$$

and symmetrize with respect to the first $2n+1$ variables:

$$\left\langle P_{2n} \middle| x^m \right\rangle \propto \sum_{k=1}^{2n+1} \int \bar{\mathrm{d}}y\ \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_{2n+1}\ \ y^m \prod_{1\leqslant i<j\leqslant 2n+1} (\lambda_i - \lambda_j)$$
$$\times \prod_{1\leqslant i<j\leqslant 2n+1} \operatorname{sgn}(\lambda_i - \lambda_j) \prod_{1\leqslant i\neq k\leqslant 2n+1} \operatorname{sgn}(\lambda_i - \lambda_k)\ \operatorname{sgn}(\lambda_k - y). \tag{3.23}$$

Note the following identity:

$$\prod_{i=1}^{2n+1} \operatorname{sgn}(y-\lambda_i) = \sum_{k=1}^{2n+1} \operatorname{sgn}(y-\lambda_k) \prod_{i=1, i\neq k}^{2n+1} \operatorname{sgn}(\lambda_k - \lambda_i) \tag{3.24}$$

which gives (and note $y = \lambda_{2n+2}$)

$$\left\langle P_{2n} \middle| x^m \right\rangle \propto \int \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_{2n+2} \left[\lambda_{2n+2}^m \prod_{1\leqslant i<j\leqslant 2n+1} (\lambda_i - \lambda_j) \right] \left[\prod_{1\leqslant i<j\leqslant 2n+2} \operatorname{sgn}(\lambda_i - \lambda_j) \right]. \tag{3.25}$$

The second bracket is completely antisymmetric in the $2n+2$ variables, so that we have to antisymmetrize the first bracket as well. The result is zero when $m \leqslant 2n$, because any non-zero antisymmetric polynomial of $2n+2$ variables must have degree at least $2n+1$, while the first bracket is a polynomial of degree at most $2n$ in any of its variables.

By the same argument, one would find that

$$\left\langle P_{2n+1} \middle| x^m \right\rangle \propto \int \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_{2n+2} \left[\lambda_{2n+2}^m \left(c_n + \sum_{i=1}^{2n+1} \lambda_i \right) \prod_{1\leqslant i<j\leqslant 2n+1} (\lambda_i - \lambda_j) \right]$$
$$\times \left[\prod_{1\leqslant i<j\leqslant 2n+2} \operatorname{sgn}(\lambda_i - \lambda_j) \right] \tag{3.26}$$

which, by antisymmetrization of the first bracket, vanishes when $m \leqslant 2n-1$.

3.2. Proof of (3.18)

Again, it is sufficient to prove that

$$\left\langle P_{2n}\middle|x^m\right\rangle = 0 \qquad \text{and} \qquad \left\langle P_{2n+1}\middle|x^m\right\rangle = 0 \qquad \text{for all} \quad m \leqslant 2n-1 \tag{3.27}$$

$$\left\langle P_{2n}\middle|x^m\right\rangle \propto \int \bar{\mathrm{d}}x\ \bar{\mathrm{d}}\lambda_1 \ldots \bar{\mathrm{d}}\lambda_n \times \prod_{1\leqslant i<j\leqslant n}(\lambda_i-\lambda_j)^4 \prod_{i=1}^n (x-\lambda_i)^2 \left(m x^{m-1} - x^m \sum_{i=1}^n \frac{2}{x-\lambda_i}\right). \tag{3.28}$$

Introduce n extra variables $(\mu_1,\ldots,\mu_n)$, and consider the $2n\times 2n$ Vandermonde determinant of the $2n$ variables (λ_i,μ_i), divide it by $\prod_i(\lambda_i-\mu_i)$ and take the limit $\mu_i\to\lambda_i$. You get

$$\prod_{1\leqslant i<j\leqslant n}(\lambda_i-\lambda_j)^4 = \lim_{\mu_i\to\lambda_i}\frac{\Delta_{2n}(\lambda_i,\mu_i)}{\prod_{i=1}^n(\lambda_i-\mu_i)} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{2n-1} \\ \vdots & & & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{2n-1} \\ 0 & 1 & 2\lambda_1 & \ldots & (2n-1)\lambda_1^{2n-2} \\ \vdots & & & & \vdots \\ 0 & 1 & 2\lambda_n & \ldots & (2n-1)\lambda_n^{2n-2} \end{pmatrix}. \tag{3.29}$$

By the same method, we have

$$\prod_{1\leqslant i<j\leqslant n}(\lambda_i-\lambda_j)^4 \prod_{i=1}^n (x-\lambda_i)^2 = \lim_{\mu_i\to\lambda_i}\frac{\Delta_{(}x,\lambda_i,\mu_i)}{\prod_{i=1}^n(\lambda_i-\mu_i)} = \det\begin{pmatrix} 1 & x & x^2 & \ldots & x^{2n} \\ 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{2n} \\ \vdots & & & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{2n} \\ 0 & 1 & 2\lambda_1 & \ldots & 2n\lambda_1^{2n-1} \\ \vdots & & & & \vdots \\ 0 & 1 & 2\lambda_n & \ldots & 2n\lambda_n^{2n-1} \end{pmatrix} \tag{3.30}$$

and

$$\frac{\partial}{\partial x}\prod_{1\leqslant i<j\leqslant n}(\lambda_i-\lambda_j)^4 \prod_{i=1}^n (x-\lambda_i)^2 = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{2n} \\ \vdots & & & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{2n} \\ 0 & 1 & 2x & \ldots & 2nx^{2n-1} \\ 0 & 1 & 2\lambda_1 & \ldots & 2n\lambda_1^{2n-1} \\ \vdots & & & & \vdots \\ 0 & 1 & 2\lambda_n & \ldots & 2n\lambda_n^{2n-1} \end{pmatrix}. \tag{3.31}$$

Therefore, the integrand in (3.28) is a $(2n+2)\times(2n+2)$ determinant

$$\det\begin{pmatrix} 1 & x & x^2 & \dots & x^{2n} & x^m \\ 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{2n} & 0 \\ \vdots & & & & \vdots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{2n} & 0 \\ 0 & 1 & 2x & \dots & 2nx^{2n-1} & mx^{m-1} \\ 0 & 1 & 2\lambda_1 & \dots & 2n\lambda_1^{2n-1} & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & 1 & 2\lambda_n & \dots & 2n\lambda_n^{2n-1} & 0 \end{pmatrix} \tag{3.32}$$

we note $x=\lambda_{n+1}$, and by antisymmetrization, it becomes

$$\det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{2n} & \lambda_1^m \\ \vdots & & & & \vdots & \vdots \\ 1 & \lambda_{n+1} & \lambda_{n+1}^2 & \dots & \lambda_{n+1}^{2n} & \lambda_{n+1}^m \\ 0 & 1 & 2\lambda_1 & \dots & 2n\lambda_1^{2n-1} & m\lambda_1^{m-1} \\ \vdots & & & & \vdots & \vdots \\ 0 & 1 & 2\lambda_{n+1} & \dots & 2n\lambda_{n+1}^{2n-1} & m\lambda_{n+1}^{m-1} \end{pmatrix} \tag{3.33}$$

which obviously vanishes when $m\leqslant 2n$.

By the same argument, one would find that $\langle P_{2n+1}|x^m\rangle$ reduces to the same kind of integral, but with m replaced by $m+1$, and vanishes when $m\leqslant 2n-1$.

We have thus proven that the skew orthogonal polynomials are indeed given by (3.16) and (3.18).

4. Large N asymptotics

The large N universal statistical properties of a random $N\times N$ matrix M belonging to one of the three ensembles $E_N^{(\beta)}$, can be expressed in terms of a few polynomials P_n, with n close to the 'Fermi level' [3, 17, 18]:

$$n_{\mathrm{F}} = \frac{\beta}{2}N_\beta. \tag{4.34}$$

More precisely, for $\beta=2$ we need asymptotics of P_n in the limit

$$N\to\infty \qquad n\to\infty \qquad n-N\sim O(1) \tag{4.35}$$

for $\beta=1$ we need asymptotics of P_{2n} and P_{2n+1} in the limit

$$N\to\infty \qquad n\to\infty \qquad 2n-N\sim O(1) \tag{4.36}$$

and for $\beta=4$ we need asymptotics of P_{2n} and P_{2n+1} in the limit

$$N\to\infty \qquad n\to\infty \qquad n-N=n-2N_4\sim O(1). \tag{4.37}$$

4.1. The resolvent

We introduce the function $W(z)$ usually called the resolvent or Green function

$$W(z) \underset{\text{def}}{=} W_m^{(\beta)}[\mathcal{V}](z) \underset{\text{def}}{=} \frac{1}{m}\left\langle \sum_{k=1}^{m} \frac{1}{z-\lambda_k} \right\rangle \propto \frac{1}{m}\left\langle \mathrm{tr}\frac{1}{z-M} \right\rangle \tag{4.38}$$

where $M \in E_m^{(\beta)}$ and the mean value is taken with respect to the weight

$$\mathrm{e}^{-m_\beta \,\mathrm{tr}\, \mathcal{V}(M)}. \tag{4.39}$$

When there is no ambiguity, we will drop the β, m or $\mathcal{V}$ indices, and write the resolvent as $W(z)$. Note that we have chosen a normalization such that

$$W(z) \underset{z\to\infty}{\sim} \frac{1}{z}. \tag{4.40}$$

The reason to introduce the resolvent is that the logarithmic derivative of $P_n(x)$ is proportional to the resolvent $W_m(z)$ (from (3.14), (3.16) and (3.18), at least when n is even) for some appropriate value of m, and with a potential of the form

$$\mathcal{V}(z) = \frac{1}{T}V(z) - r\,\ln(x-z). \tag{4.41}$$

More precisely, we have:

- In the unitary case $\beta = 2$:

$$\frac{P_n^{(2)'}(x)}{P_n^{(2)}(x)} = n\,W_n(z)|_{z=x} \qquad \text{with} \qquad \mathcal{V}(z) = \frac{N}{n}V(z) - \frac{1}{n}\ln(x-z) \tag{4.42}$$

  i.e. $m = n$, $r = \frac{1}{n}$ and $T = \frac{n}{N}$ ($\to 1$ when $n \to n_{\mathrm{F}}$).
- In the orthogonal case $\beta = 1$:

$$\frac{P_{2n}^{(1)'}(x)}{P_{2n}^{(1)}(x)} = 2n\,W_{2n}(z)|_{z=x} \qquad \text{with} \qquad \mathcal{V}(x) = \frac{N}{m}V - \frac{1}{m}\ln(x-z) \tag{4.43}$$

  i.e. $m = 2n$, $r = \frac{1}{2n}$ and $T = \frac{2n}{N}$ ($\to 1$ when $n \to n_{\mathrm{F}}$).
- In the symplectic case $\beta = 4$:

$$\frac{P_{2n}^{(4)'}(x)}{P_{2n}^{(4)}(x)} = 2n\,W_n(z)|_{z=x} \qquad \text{with} \qquad \mathcal{V}(x) = \frac{N}{n}V - \frac{2}{n}\ln(x-z) \tag{4.44}$$

  i.e. $m = n$, $r = \frac{2}{n}$ and $T = \frac{n}{N}$ ($\to 1$ when $n \to n_{\mathrm{F}}$).

  In all three cases $T = \frac{n}{n_{\mathrm{F}}}$ and $r = \frac{\beta}{2n}$.

4.2. Asymptotics for the resolvent

In a potential $\mathcal{V}$, the resolvent $W(z) = W_m(z)$ satisfies the equations of motion (resulting from invariance of an integral like equation 2.1 under a change of variable $M \to f(M)$):

$$W(z)^2 - \frac{\eta}{2n}W'(z) = \frac{2}{\beta}\mathcal{V}'(z)W(z) - Q(z) + O(1/n^2) \tag{4.45}$$

where $\eta = (1, 0, -1)$ respectively for $\beta = (1, 2, 4)$ and $Q(z)$ is a polynomial[3] of degree deg $V - 2$, which is not determined by the equations of motions and has to be determined by analytical considerations, for instance the one-cut assumption.

[3] When $\mathcal{V}'$ has poles, Q may have poles too. $Q(z)$ is a rational function, whose poles must be chosen in order to cancel the poles of $W(z)$ in equation 4.45.

Here, we will consider a potential $\mathcal{V}$ of the form

$$\mathcal{V}(z) = \frac{1}{T} V(z) - r \ln(x - z) \tag{4.46}$$

and we will be interested in the limit where $T - 1$ and r are small of order $1/n$.

The method is to find first the solution $W(z)$ at $T = 1$ and $r = 0$. We write it

$$W(z) = W_0(z) + \frac{\eta}{2n} W_1(z) + O(1/n^2) \tag{4.47}$$

and then, add the variations

$$(T - 1)\frac{\partial}{\partial T} W_0 + r\frac{\partial}{\partial r} W_0 \tag{4.48}$$

(to order $1/n$, we do not need to consider the variations of W_1 with respect to T and r), the derivatives are taken at $T = 1$ and $r = 0$.

4.3. *Contribution of W_0*

The function $W_0(z)$, (as well as its derivatives with respect to T and r) has been extensively studied in RMT. Note that W_0 is nearly the same for $\beta = 1$, 2 or 4. Let us recall here some of the main features of W_0 in order to fix the notations.

At $n \to \infty$ ($T = 1$ and $r = 0$), 4.45 reduces to a quadratic equation for $W_0(z)$. The one-cut solution is

$$W_0(z) = \frac{1}{\beta}\left(V'(z) - M(z)\sqrt{(z-a)(z-b)}\right) \tag{4.49}$$

where $M(z)$ is a polynomial of degree $d - 1$ ($d = \deg V'$), which is completely determined by the large z limit condition 4.40

$$M(z) = \operatorname*{Pol}_{z\to\infty} \frac{V'(z)}{z\sqrt{(1-a/z)(1-b/z)}}. \tag{4.50}$$

The end-points a and b too are determined by (4.40) which implies

$$\oint \frac{V'(z)}{\sqrt{(z-a)(z-b)}}\,\mathrm{d}z = 0 \qquad \oint \frac{zV'(z)}{\sqrt{(z-a)(z-b)}}\,\mathrm{d}z = 2\mathrm{i}\pi\beta \tag{4.51}$$

where the contour encircles the cut $[a, b]$ in the counterclockwise direction.

We also introduce the function $\rho(z) = \frac{1}{\beta\pi} M(z)\sqrt{(z-a)(b-z)}$, defined for z complex. It is such that

$$W_0(z) = \frac{1}{\beta} V'(z) - \mathrm{i}\pi\rho(z). \tag{4.52}$$

When $z \in [a, b]$, $\rho(z)$ is real and coincides with the average density of eigenvalues of the random matrix in the large N limit[4]. Indeed from (4.38), $W(z) = \int_a^b \frac{1}{z-\lambda}\rho(\lambda)\,\mathrm{d}\lambda$. Note that (4.40) implies that the density is normalized:

$$\int_a^b \mathrm{d}z\,\rho(z) = 1. \tag{4.53}$$

It is useful to notice that $W_0(z)$ obeys a linear Riemann–Hilbert type equation:

$$W_0(z + \mathrm{i}0) + W_0(z - \mathrm{i}0) = \frac{2}{\beta} V'(z) \qquad \text{when} \quad z \in [a, b]. \tag{4.54}$$

[4] As an example, consider the Gaussian case: V is quadratic, i.e. V' is of degree $d = 1$, thus $M(z)$ is a constant and $\rho(z) = \sqrt{(z-a)(b-z)}$ is the famous Wigner's semi-circle law.

Some notations. It will be convenient to parametrize z as

$$z = \frac{a+b}{2} + \frac{b-a}{2}\cos\phi(z) = \frac{a+b}{2} + 2\alpha\,\cos\phi(z) \qquad \text{where} \quad \alpha = \frac{b-a}{4} \tag{4.55}$$

$\phi(z)$ is defined for all complex z and is a multi-valued function. We will see that both determinations $\phi(z)$ and $-\phi(z)$ will enter the asymptotic expression of the orthogonal polynomials when $z \in [a, b]$.

We also define $\sigma(z)$ as

$$\sigma(z) = (z-a)(z-b). \tag{4.56}$$

We have

$$\sqrt{\sigma(z)} = 2\mathrm{i}\alpha\,\sin\phi \qquad \text{and} \qquad \mathrm{i}\phi'(z) = \frac{1}{\sqrt{\sigma(z)}}. \tag{4.57}$$

4.4. Variations of W_0 with respect to T and r

It can be proven from (4.54) and (4.40) (see [13] for instance) that

$$W_T(z) \underset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}T} T\, W_0(z) = \frac{1}{\sqrt{\sigma(z)}} = \mathrm{i}\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} \tag{4.58}$$

and

$$W_r(z) \underset{\text{def}}{=} \beta\frac{\mathrm{d}}{\mathrm{d}r} W_0(z) = -\frac{1}{\sqrt{\sigma(z)}}\frac{\sqrt{\sigma(z)} - \sqrt{\sigma(x)}}{(z-x)} + \frac{1}{\sqrt{\sigma(z)}}. \tag{4.59}$$

In particular at $z = x$, we have

$$W_r(x) = -\frac{\sigma'(x)}{2\sigma(x)} + \frac{1}{\sqrt{\sigma(x)}}. \tag{4.60}$$

4.5. Contribution of W_1

For $T = 1$ and $r = 0$, and to order $O(1/n)$, the equation of motion reduces to

$$W^2(z) - \frac{\eta}{2n} W'(z) + O(1/n^2) = \frac{2}{\beta} V'(z) W(z) - Q(z) \tag{4.61}$$

and we expand $W(z)$ to first order in $1/n$ as

$$W(z) \sim W_0(z) + \frac{\eta}{2n} W_1(z) + O(1/n^2). \tag{4.62}$$

To order $1/n$, equation (4.61) gives (using the value of $W_0(z)$ from (4.49)):

$$\frac{2}{\beta} W_1(z) = \frac{Q_1(z) - W_0'(z)}{M(z)\sqrt{\sigma(z)}} \tag{4.63}$$

where $Q_1(z)$ is a polynomial of degree $d-2$.

Let us factorize $M(z)$ (recall that $d = \deg V'$ and g is the leading coefficient of V'):

$$M(z) = g \prod_{k=1}^{d-1} (z - z_k) \tag{4.64}$$

and decompose W_1 into single pole terms.The condition that $W_1(z)$ is regular when $z = z_k$ determines the polynomial $Q_1(z)$, we obtain

$$W_1(z) = \frac{\sigma'(z)}{4\sigma(z)} + \frac{1}{2}\sum_{k=1}^{d-1} \frac{\sqrt{\sigma(z)} - \sqrt{\sigma(z_k)}}{(z-z_k)\sqrt{\sigma(z)}} - \frac{d}{2\sqrt{\sigma(z)}}. \tag{4.65}$$

With the parameterization $z = \frac{a+b}{2} + 2\alpha\cos\phi$ and $z_k = \frac{a+b}{2} + 2\alpha\cos\phi_k$, we have

$$W_1(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{4}\ln\sigma(z) + \sum_{k=1}^{d-1}\ln\sin\left(\frac{\phi(z)+\phi_k}{2}\right) - \frac{d}{2}\mathrm{i}\phi(z)\right]. \tag{4.66}$$

4.6. Asymptotics of the skew orthogonal polynomials

We have computed all the contributions to the asymptotics of the resolvent

$$W(z) \sim \frac{1}{T}W_0(z) + \frac{T-1}{T}W_T(z) + \frac{r}{\beta}W_r(z)\frac{\eta}{2n}W_1(z) + O(1/n^2) \tag{4.67}$$

i.e.

$$2nW(z) \sim \beta N_\beta W_0(z) + (2n-\beta N_\beta)W_T(z) + W_r(z) + \eta W_1(z) + O(1/n) \tag{4.68}$$

where W_0, W_T, W_r, W_1 are given by (4.52), (4.58), (4.59) (or (4.60)), (4.65) (or (4.66)).

Combining everything together:

- $\beta = 2$. From $P_n'/P_n = nW(x)$, we get the asymptotic behaviour of the orthogonal polynomials (already known [2,13,14]):

$$P_n^{(2)}(x)\mathrm{e}^{-\frac{N}{2}V(x)} \sim \frac{C_n^{(2)}}{\sqrt{2\mathrm{i}\alpha\sin\phi}}\,\mathrm{e}^{-N\mathrm{i}\pi\int_a^x\rho(y)\,\mathrm{d}y}\mathrm{e}^{\mathrm{i}(n-N+\frac{1}{2})\phi} + \text{c.c.} \tag{4.69}$$

The normalization constant $C_n^{(2)} = \alpha^{n+\frac{1}{2}}$ is such that $P_n(x) \sim x^n$ for large x. Equation (4.69) is basically the contribution of W_0, which is the same for all three cases $\beta = 1, 2, 4$. The $\beta = 1$ and $\beta = 4$ cases contain an extra contribution from W_1.

- $\beta = 1$. From $P_{2n}'/P_{2n} = 2nW(x)$ we get.

$$P_{2n}^{(1)}(x)\mathrm{e}^{-NV(x)} \sim \frac{C_n^{(1)}}{\sqrt{2\mathrm{i}\alpha\sin\phi}}\,\mathrm{e}^{-N\mathrm{i}\pi\int_a^x\rho(y)\,\mathrm{d}y}\,\mathrm{e}^{\mathrm{i}(2n+1-N-\frac{d}{2})\phi}\,M_+(\phi) + \text{c.c.} \tag{4.70}$$

where

$$M_+(\phi) = M_-(-\phi) = \prod_{k=1}^{d-1} 2\mathrm{i}\sin\left(\frac{\phi+\phi_k}{2}\right). \tag{4.71}$$

Note that $M(x) = g\alpha^{d-1}M_+(\phi)M_-(\phi)$, where g is the leading coefficient of $V'(x)$.

$$C_n^{(1)} = \alpha^{2n+\frac{1}{2}}\prod_{k=1}^{d-1}\mathrm{e}^{-\mathrm{i}\phi_k/2} \tag{4.72}$$

is the normalization constant chosen so that $P_{2n}(x) \sim x^{2n}$ for large x.

The odd polynomial is found from $P_{2n+1}/P_{2n} = \langle x + \mathrm{tr}M + c_n\rangle$ and $\langle\mathrm{tr}M\rangle = 2n\lim_{z\to\infty} z^2(W(z)-1)$ (note that we need (4.59), not (4.60)). The whole x dependence of P_{2n+1}/P_{2n} comes from $x + \lim_{z\to\infty} z^2(W_r(z)-1) = \sqrt{\sigma(x)} - \frac{a+b}{2}$. Therefore, (and up to an arbitrary linear combination with P_{2n}), we have

$$P_{2n+1}^{(1)}(x)\mathrm{e}^{-NV(x)} \sim C_n^{(1)}\sqrt{2\mathrm{i}\alpha\sin\phi}\,\mathrm{e}^{-N\mathrm{i}\pi\int_a^x\rho(y)\,\mathrm{d}y}\,\mathrm{e}^{\mathrm{i}(2n+1-N-\frac{d}{2})\phi}\,M_+(\phi) + \text{c.c.} \tag{4.73}$$

- $\beta = 4$. From $P'_{2n}/P_{2n} = 2nW(x)$ we get

$$P^{(4)}_{2n}(x)\mathrm{e}^{-\frac{N}{2}V(x)} \sim \frac{C^{(4)}_n}{\sqrt{2\mathrm{i}\alpha\ \sin\phi}}\ \mathrm{e}^{-2N\mathrm{i}\pi\int_a^x \rho(y)\,\mathrm{d}y}\ \mathrm{e}^{\mathrm{i}(2n+1-2N+\frac{d}{2})\phi}\ \frac{M_-(\phi)}{i\rho(x)} + \text{c.c.} \tag{4.74}$$

with normalization constant

$$C^{(4)}_n = \frac{g}{4\pi}\alpha^{2n+d+\frac{1}{2}}\prod_{k=1}^{d-1}\mathrm{e}^{\mathrm{i}\phi_k/2} \tag{4.75}$$

and

$$P^{(4)}_{2n+1}(x)\mathrm{e}^{-\frac{N}{2}V(x)} \sim C^{(4)}_n\sqrt{2\mathrm{i}\alpha\ \sin\phi}\,\mathrm{e}^{-2N\mathrm{i}\pi\int_a^x \rho(y)\,\mathrm{d}y}\ \mathrm{e}^{\mathrm{i}(2n+1-2N+\frac{d}{2})\phi}\ \frac{M_-(\phi)}{i\rho(x)} + \text{c.c.} \tag{4.76}$$

Note that we have used $\mathrm{i}\rho(x) = \frac{g}{4\pi}\alpha^d\ M_+(\phi)\ M_-(\phi)\ 2\mathrm{i}\sin\phi$.

Some remarks

- The derivation presented here is actually valid only when $x \notin [a, b]$, giving only one exponential term, with the determination of $\rho(x)$ and $\phi(x)$ (from (4.55)) such that $P_n(x)\mathrm{e}^{-NV(x)}$ decreases when $x \to \infty$. When $x \in [a, b]$, a careful analysis shows that both determinations of $\phi(x)$ must be taken into account. The only effect is to add the complex conjugate exponential (c.c.) to the asymptotics, so that $P_n(x)$ is indeed real when $x \in [a, b]$. Outside $[a, b]$, $P_n\mathrm{e}^{-NV}$ decreases exponentially, and in $[a, b]$, it oscillates like a cosine function, and it indeed has n zeroes.
- The derivation was carried out only in the 'one-cut' case. It was assumed that the support of the density of eigenvalues (for $N \to \infty$) is connected and is made of one interval $[a, b]$.
- The derived asymptotics are not valid when x is close to the end-points a or b. One must have $|(x - a)(x - b)| > O(N^{-\gamma})$, where γ is some positive (and rational) exponent which depends on the potential V. For generic[5] V, (in particular for V quadratic), we have $\gamma = 2/3$ [20, 21].
- Note that the above expressions all have the correct large x behaviour $P_n(x) \sim x^n$. It can be seen easily if one remembers that $x \sim \alpha\mathrm{e}^{\mathrm{i}\phi}$ when $\mathrm{i}\phi \to +\infty$.

4.7. Check of orthogonality

We have presented a derivation of the asymptotics (4.70)–(4.76), so that there should be no reason to doubt they fulfil the orthogonality condition. However, it is interesting to see how. We will just sketch the procedure.

In all cases, we have to compute integrals of $P_nP_m\mathrm{e}^{-NV}$, with x running from $-\infty$ to $+\infty$. The contributions outside $[a,b]$ are exponentially small, the integrals can thus be computed inside $[a, b]$. Within $[a, b]$, terms which oscillate exponentially fast like $\mathrm{e}^{Ni\pi\int\rho}$, average to zero to order $O(1/N)$, so that to leading order, it is sufficient to consider only the cross-terms in the product P_nP_m, with opposite signs for the two determinations of ϕ.

In the $\beta = 1$ case, the scalar product $\langle P_n|P_m\rangle$ of (2.7) can be computed by integration by parts. For that, you need a primitive of $P_n\mathrm{e}^{-NV}$, which is achieved to leading order by dividing (4.70) or (4.73) by $\rho(x) \propto M_+(\phi)M_-(\phi)\sin\phi$.

[5] γ depends on m_a and m_b where $\rho(z) \sim (z - a)^{m_a+1/2}$ near $z = a$ and $\rho(z) \sim (z - b)^{m_b+1/2}$ near $z = b$. When $m_a = m_b = 0$, we have $\gamma = 2/3$.

In the $\beta = 4$ case, you need a derivative of $P_n \mathrm{e}^{-\frac{N}{2}V}$, which is achieved to leading order by multiplying (4.74) or (4.76) by $\rho(x) \propto M_+(\phi)M_-(\phi)\sin\phi$.

Then you find that in both cases ($\beta = 1$ and 4), and up to unimportant constant factors, you have to leading order (up to $O(1/n)$)

$$\langle P_{2n}|P_{2m}\rangle \propto \int_0^\pi \mathrm{d}\phi\ \frac{\sin 2(n-m)\phi}{\sin\phi} = 0 \tag{4.77}$$

$$\langle P_{2n+1}|P_{2m+1}\rangle \propto \int_0^\pi \mathrm{d}\phi\ \ \sin\phi\ \sin 2(n-m)\phi = 0 \tag{4.78}$$

$$\langle P_{2n+1}|P_{2m}\rangle \propto \int_0^\pi \mathrm{d}\phi\ \ \cos 2(n-m)\phi \propto\ \delta_{nm} \tag{4.79}$$

which confirms that our asymptotics indeed fulfil the orthogonality properties.

Taking into account properly the constant factors, we can determine the h_n's:

- $\beta = 2$:
$$h_n^{(2)} \sim 2\pi\ \alpha^{2n+1} \tag{4.80}$$
- $\beta = 1$:
$$h_n^{(1)} \sim \frac{16\pi}{Ng\alpha^{d+1}}\ \alpha^{4n+3} \tag{4.81}$$
- $\beta = 4$:
$$h_n^{(4)} \sim 2N\pi g\alpha^{d+1}\ \alpha^{4n+1}. \tag{4.82}$$

5. Conclusions

Therefore, we have obtained some exact integral expressions and asymptotics for the skew orthogonal polynomials involved in the orthogonal and symplectic random matrix ensembles.

Our asymptotics were derived in the 'one-cut' case only, though it seems likely that the result could be extended easily to the multicut case, following the method of [15] or [16] and would involve hyper-elliptical theta functions instead of exponentials.

Another possible extension of the method presented here is to 'multi-matrix models', and a time-dependent matrix, as in [13]. It seems that the same kind of asymptotics could be obtained.

The asymptotics of the skew orthogonal polynomials are useful to evaluate the Christoffel–Darboux kernels

$$K(\lambda,\mu) = \frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{h_n}\left(P_{2n}(\lambda)P_{2n+1}(\mu) - P_{2n+1}(\lambda)P_{2n}(\mu)\right)\,\mathrm{e}^{-NV(\lambda)}\,\mathrm{e}^{-NV(\mu)} \tag{5.83}$$

which give all the correlation functions. For instance with $\beta = 4$, we have [3]

$$\rho(\lambda) = -\frac{\partial}{\partial\lambda}K(\lambda,\mu)\bigg|_{\mu=\lambda} \tag{5.84}$$

$$\rho_c(\lambda,\mu) = -\frac{\partial}{\partial\lambda}K(\lambda,\mu)\ \frac{\partial}{\partial\mu}K(\lambda,\mu) + K(\lambda,\mu)\ \frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}K(\lambda,\mu). \tag{5.85}$$

In order to use the asymptotics of the orthogonal polynomials in (5.83), one needs a generalization of the Christoffel–Darboux theorem, which yields $K(\lambda,\mu)$ in terms of a few

P_n only with n close to the Fermi level n_F. With asymptotics of the type (4.69), (4.70), (4.73), (4.74) or (4.76), the Christoffel–Darboux theorem [17, 18] merely amounts to a formal resummation of the geometrical series (it was proven in [13] for hermitian multi-matrix models, and the same proof would work here). For instance in the $\beta = 4$ case, the generalization of the Christoffel–Darboux theorem reads

$$\sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{i}(2n+3-2N)(\phi(\lambda)-\phi(\mu))} \sim \frac{1}{2i\,\sin(\phi(\lambda)-\phi(\mu))}. \tag{5.86}$$

This trick yields asymptotics for the kernels $K(\lambda,\mu)$, and subsequently asymptotics for all the correlation functions. One can then easily check that in the short distance regime $|\lambda-\mu| \sim O(1/N)$, the universal 2-point connected correlation function is well reproduced, and that in the long distance regime $|\lambda-\mu| \sim O(1)$, the smoothed 2-point connected correlation function is correctly reproduced too. The leading behaviour of short- and long-distance correlation functions was already known from other methods [3], so that our method does not provide any new result for the correlation functions. However, it seems that our asymptotics could be used to build a rigorous mathematical proof of the universality, following the method of [10], because they allow a good control of the approximations.

In addition, the fact that the skew orthogonal polynomials are exactly the average characteristic polynomials of the random matrices is remarkable. It would be interesting to understand the generality of this result, and for instance try to generalize it to the other random matrix ensembles related to Cartan's classification of symmetric spaces [19].

Appendix A1. The symplectic ensemble $\beta = 4$

$E_N^{(4)}$ is the set of all real-quaternion-self-dual matrices M, of size $2N \times 2N$.

One can view a $2N \times 2N$ matrix M as a block matrix with N^2 blocks of size 2×2

$$\tilde{M}_{ij} = \begin{pmatrix} M_{2i-1,2j-1} & M_{2i,2j-1} \\ M_{2i-1,2j} & M_{2i,2j} \end{pmatrix}.$$

By definition, $M \in E_N^{(4)}$ means that each $\tilde{M}_{ij}$ is a real-quaternion (see appendix A2). The matrix $(\tilde{M})_{ij}$ $(1 \leqslant i, j \leqslant N)$ is a $N \times N$ matrix with real-quaternion entries, and self duality means that $\tilde{M}_{ij} = \overline{\tilde{M}}_{ji}$, which implies that M is hermitian: $M^\dagger = M$.

Note that $ZMZ = -M^t$ where $Z = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \otimes \mathbf{Id}_N$.

M is diagonalizable (by a symplectic transformation) and all its eigenvalues are degenerated twice. Let $(\lambda_1, \ldots, \lambda_N)$ be the eigenvalues. The trace and the determinant of M are

$$\mathrm{tr} M = 2\sum_{j=1}^{N} \lambda_j \qquad \det M = (\,\mathrm{Pf}\,\tilde{M}\,)^2 = \left(\prod_{j=1}^{N} \lambda_j\right)^2. \tag{A1.87}$$

The measure $\mathrm{d}M$ on $E_N^{(4)}$ is

$$\mathrm{d}M \underset{\mathrm{def}}{=} \prod_{i=1}^{N} \mathrm{d}\tilde{M}_{ii}^{(0)} \prod_{1\leqslant i<j\leqslant N} \mathrm{d}\tilde{M}_{ij}^{(0)}\,\mathrm{d}\tilde{M}_{ij}^{(1)}\,\mathrm{d}\tilde{M}_{ij}^{(2)}\,\mathrm{d}\tilde{M}_{ij}^{(3)}. \tag{A1.88}$$

Appendix A2. Quaternions

A real-quaternion q can be represented as a 2×2 matrix of the following form:

$$q = q^{(0)}\mathbf{I}_2 + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3 \tag{A2.89}$$

where $q^{(0)}$, $q^{(1)}$, $q^{(2)}$, $q^{(3)}$ are real numbers and

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{A2.90}$$

The conjugate of a quaternion q is

$$\overline{q} = q^{(0)}\mathbf{I}_2 - q^{(1)}e_1 - q^{(2)}e_2 - q^{(3)}e_3. \tag{A2.91}$$

The set of quaternions is a non-commutative field (note that any non-zero quaternion is invertible).

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